Solution to Exercise 6

1. Please refer to Lemma 2.2 in Note 1.

2. Since $K \subset \mathrm{ri}(\mathrm{dom}\, f)$, we can assume without loss of generality that K is convex. Otherwise, we can find some compact convex K' containing K s.t. $K' \subset \mathrm{ri}(\mathrm{dom}\, f)$ and we just need to prove the global Lipschitz continuity on K'.

Using local Lipschitz continuity, $\forall x \in K$, there is some open ball $B_x \subset \mathrm{dom}\, f$ s.t. f is local Lipschitz on B_x Then $\{B_x\}_{x\in K}$ is an open cover of K. Since K is compact, there exists a finite subcover $\{B_{x_n}\}$. Then letting M be the maximum Lipschitz constant on $\{B_{x_n}\}$, it follows that $|f(x) - f(y)| \leq M\|x - y\|_2$, $\forall x, y \in K$.

3. (1) Consider $f(x) = \frac{1}{x}$, $\mathrm{dom}\, f = (0, +\infty), K = (0, 1]$. Since $f'(x) = -\frac{1}{x^2}$ is unbounded on K, f is not Lipschitz continuous.

(2) Consider $f(x) = x^2$, $\mathrm{dom}\, f = K = \mathbb{R}$. Since $f'(x) = 2x$ is unbounded on $K = \mathbb{R}$, the function is not Lipschitz continuous on K.

(3) Consider $f(x) = -\sqrt{x}$, $dom f = [0, +\infty), K = [0, 1]$. Since $\lim_{x\to 0+} \frac{f(x)-f(0)}{x} = \lim_{x\to 0+} \frac{-1}{\sqrt{x}}$ does not exist, f is not Lipschitz continuous on K.